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B.Sc. (Physics) Part-I, Paper-I, Group-B

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Lagrange's Equations:

Here we obtain the equations of motion for a single particle in terms of generalized coordinates. This gives us the Lagrange's equations. In case of Lagrangian dynamics we need to know the energy of the particle rather than force (concept of force is used to formulate Newtonian dynamics.

Let us consider that T is the kinetic energy which is given by (in cartesian coordinate system (x, y, z))

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{— (1)}$$

Let $q_1, q_2, \ldots q_n$ denotes the generalized coordinates

$$x = x(q_1, q_2, \ldots q_n) = x(q) \quad \text{— (2)}$$

Similarly $y = y(q)$, $z = z(q)$ — (3)

From (2)

$$\dot{x} = \frac{\partial x}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2}\frac{\partial q_2}{\partial t} + \cdots + \frac{\partial x}{\partial q_n}\frac{\partial q_n}{\partial t}$$

$$= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\frac{\partial q_k}{\partial t} = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\dot{q}_k = \dot{x}(q, \dot{q}) \quad \text{— (4)}$$

$\dot{q} \rightarrow$ generalized velocity: component of velocity in Cartesian coordinates can be expressed in terms of generalized coordinates and generalized velocity as given by Eq. (4).

Now we can write $\dot{x} = \dot{x}(q, \dot{q}), \dot{y} = \dot{y}(q, \dot{q}), \dot{z} = \dot{z}(q, \dot{q})$ — (5)

Now using Eq. (5) in Eq. (1),

$$T = \frac{1}{2} m \left[\dot{x}^2(q,\dot{q}) + \dot{y}^2(q,\dot{q}) + \dot{z}^2(q,\dot{q}) \right] \quad \text{—— (6)}$$

Now taking derivative w.r.to $\dot{q}_k$.

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_k} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_k} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_k} \right) \quad \text{—— (7)}$$

From Eq. (4), we can write. $\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k}$ —— (8)

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial x}{\partial q_k} + \dot{y}\frac{\partial y}{\partial q_k} + \dot{z}\frac{\partial z}{\partial q_k} \right)$$

Again differentiating both sides w.r.t. t, we obtain.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{x}\frac{\partial x}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\ddot{y}\frac{\partial y}{\partial q_k} + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right)$$

or
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right) \quad \text{—— (9)}$$

using the fact that $\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dx}{dt}\right)$, etc. we write

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = \frac{\partial \dot{x}}{\partial q_k}$$

Now $$m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = m\dot{x}\frac{\partial \dot{x}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{x}^2\right) \quad \text{--- (10)}$$

We define $$F_x = m\ddot{x}, \quad F_y = m\ddot{y}, \quad F_z = m\ddot{z} \quad \text{--- (11)}$$

using (10) and (11) in (9), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left[\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)\right] \quad \text{--- (12)}$$

Since the generalized force is defined by

$$Q_k = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} \quad \text{--- (13)}$$

and Kinetic energy $$T = \frac{1}{2}m\left[\dot{x}^2(q,\dot{q}) + \dot{y}^2(q,\dot{q}) + \dot{z}^2(q,\dot{q})\right] \quad \text{--- (14)}$$

We write Eq. (12) as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

or $$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k} \quad \text{--- (15)}$$

Eq. (15) is the Lagrange's equations for a general standard system.

Conservative system:

When general standard system is also conservative, then we can write generalized force Q_k in terms of potential energy $V(q)$.

Let a conservative system is represented by a potential $V = V(x, y, z)$.

Force components on the particle is given by

$$F_x = \frac{-\partial V}{\partial x}, \quad F_y = \frac{-\partial V}{\partial y}, \quad F_z = \frac{-\partial V}{\partial z} \quad \text{—— (16)}$$

Since $\quad Q_k = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k}$

Thus, from (16)

$$Q_k = -\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial q_k} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_k} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_k}\right)$$

R.H.S. of above equation can be written as

$\frac{\partial V}{\partial q_k}$ ——> Here $V \equiv V(q)$

$$\therefore \quad \boxed{Q_k = -\frac{\partial V}{\partial q_k}} \quad \text{—— (17)}$$

Eq. (17) represents a conservative system.

Now using Eq. (17) in (15)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} \quad \text{—— (18)}$$

Let us define, Lagrangian function $L = T - V$

or $\quad L(q, \dot{q}) = T(q, \dot{q}) - V(q)$

Since $\frac{\partial V}{\partial \dot{q}_k} = 0$, for $V = V(q)$.

Therefore, $\quad \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k}$

From (18)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0} \quad \text{—— (19)}$$

Lagrange's equations for a conservative standard system.